SPRING 2024: MATH 590 HOMEWORK

The page and section numbers in the assignments below refer to those in the course textbook.

Wednesday, January 17. Section 1.1: 1, 3, 5, 6, 7, 12.

Friday, January 20. Section 1.2: 3, 13, 15 and Section 1.3: 1a, 1c, 3, 11a. **For a possible 2 points added to your quiz grade**: Use facts from a first course in linear algebra to prove that if $W \subsetneq \mathbb{R}^2$ is a non-zero proper subspace, then W is a line through the origin. Turn this in on Monday, January 22.

Wednesday, January 24. Section 1.3: 1, 7, 9, 10, 11b.

Friday, January 26. Section 1.4: 1(a) - 1(f), 4, 5, 7, 12a.

Monday, January 29. Section 1.4: 9a, 10a, and the following problem. Determine if the vectors $v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix}$ are linearly dependent or linearly independent. If they are linearly dependent, provide a non-trivial dependence relation among them.

Wednesday, January 31. Section 1.6: 2a, 2b, 2d, 9a, 12.

Friday, February 2. Section 1.6: 7, 9b, 15.

Monday, February 5. Section 2.1: 3.

Wednesday, February 7. Section 2.2: 3, 7, 10, 12 for 2×2 matrices only, 13 for 2×2 matrices only.

Friday, February 9. Section 2.2: 3, 5, 9 and the following problem which will appear n Quiz 4. Suppose $P_2(\mathbb{R})$ is the vector space of real polynomials of degree less than or equal to two. Let $v = 1 + 2x - 3x^2$ and $u = 2-x+5x^2$ and write $\alpha = \{1, x, x^2\}$ for the standard basis of $P_2(\mathbb{R})$. Verify that $[2v+3u]_\alpha = 2[v]_\alpha + 3[u]_\alpha$.

Monday, February 12. 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(x, y) = (x - 2y, x + 2y, 3x - 4y)$. Let $E := \{(1, -1), (1, 2)\}$ and $F := \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be bases for $\mathbb{R}^2$ and $\mathbb{R}^3$. Verify the formula $[T(v)]_F = [T]_E^F \cdot [v]_E$, for $v = (3, 2)$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (y, x)$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S(x, y) = (x + 7, x - y)$. Let $\alpha := \{(-1, 1), (0, 1)\}$, $\beta := \{(1, 0), (1, 1)\}$, and $\gamma := \{(0, -1), (-2, 0)\}$ be bases for $\mathbb{R}^2$. Verify that $[ST]_\alpha^\gamma = [S]_\beta^\gamma \cdot [T]_\alpha^\beta$.

Wednesday, February 14. Section 2.7: 1,2 and for each of these problems, verify the change of basis theorem, using the given bases.

Friday, February 16. Section 2.3: 1b, 1c, 1e, 3a, 3d, 5 (do the case $n = 4$). **And as a reminder of something to to always keep in mind**: Let $A = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$, and write C_1, C_2, C_3 for the columns of A. Verify that $A \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = uC_1 + vC_2 + wC_3$.

Monday, February 19. Section 2.3: 1f (try the case $n = 2$ first), 4 (but only for 1b, 1c), 7a, 7b (think about the $n = 2$ and $m = 3$ case).

Monday, February 26. Calculate the determinants of the matrices given in Section 3.2a: 1(a) and 1(f) in three ways: expanding along the second row, expanding along the third column, using elementary row operations to reduce to an upper triangular matrix.

Bonus Problem 2. For up to five bonus points, use the formula for expanding determinants along a row to prove the row properties (i)-(v) for **3×3 matrices** as listed in the Daily Update of February 16.

Wednesday, February 28. Section 3.3: 1, 2, 7a, 7b, 9, 10. For 9 and 10 prove the statements only for 3×3 matrices.

Friday March 1. Chapter 3, Supplementary Exercise 9a, and the following problems:

(i) Verify $|AB| = |A| \cdot |B|$, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$.

(ii) Find an orthonormal basis consisting of eigenvectors for the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Be sure to check that your basis is orthonormal.

Monday, March 4. Section 4.5: 1, 2, 3d.

Wednesday, March 6. Section 4.5: 7a,b, d for 2×2 matrices. Also: Use the definition of a symmetric linear transformation to show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 2y, 2x + y)$ is symmetric.

Friday, March 8. Section 4.1: 1a, 1c, 2a, 2b, 2c, 3c, 3d.

Monday, March 18. Section 4.1: 3a, 3c, 3f, 4, 5 (for 3×3 matrices).

Wednesday, March 20. Section 4.2: 1a, 1b, 1c, 1d, 3.

Friday, March 22. Section 4.2: 1e, 1f, 6a, 7.

Monday, March 25. Section 4.3: 2, 9a, 10a, 10b.

Bonus Problem 4. Let V denote the vector space of 2×2 matrices over $\mathbb{R}$ and define $T : V \to V$ by $T(A) = A^t$. Show that T is diagnonalizable and find a basis α for V such that the matrix of T with respect to α is diagonal. **Your solution needs to be turned in at the start of class on Friday of this week, or have the receptionist in Snow 405 put your solution in my mailbox no later than 3pm on Friday.** (5 points)

Wednesday March 27. Let V denote the three dimensional vector space of real polynomials having degree less than or equal to two with inner product $\langle f(x), g(x) \rangle := \int_{-1}^{1} f(x)g(x)\, dx$. Verify that $f_1 := \frac{1}{\sqrt{2}}$, $f_2 := \sqrt{\frac{3}{2}}x$, $f_3 := \sqrt{\frac{5}{8}}(3x^2 - 1)$ is an orthonormal basis for V and then write $p(x) = 1 + x + x^2$ in terms of this basis.

Friday, March 29. Section 4.4: 5a, 6 and the following problem. Let

$$v_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, v_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, v_3 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

be linearly independent vectors in the space of 2×2 real matrices with inner product $\langle A, B \rangle := \text{trace}(A^t B)$. Find an orthonormal basis for Span$\{v_1, v_2, v_3\}$.

Monday, April 1. Section 4.4: 1, and the following problem. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Find the QR factorization of A by setting Q to be the 3×3 matrix whose columns u_1, u_2, u_3 form the orthonormal basis for $\mathbb{R}^3$ obtained by applying the Gram-Schmidt process to the columns of A, and $R = \begin{pmatrix} C_1 \cdot u_1 & C_2 \cdot u_1 & C_3 \cdot u_1 \\ 0 & C_2 \cdot u_2 & C_3 \cdot u_2 \\ 0 & 0 & C_3 \cdot u_3 \end{pmatrix}$, where C_1, C_2, C_3 are the columns of A. Be sure to check that $A = QR$.

Monday, April 8. Verify the following properties for the matrix $A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$: (i) $\text{rank}(A^tA) = \text{rank}(AA^t)$; (ii) The eigenvalues of A^tA and AA^t are non-negative real numbers; (iii) A^tA and AA^t have the same **non-zero** eigenvalues with the same multiplicities. These are crucial observations for the development of the *Singular Value Theorem*.

Wednesday, April 10. Find the singular value decomposition of $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ by following the steps below:

1. Calculate A^tA and its characteristic polynomial $p_{A^tA}(x)$.

2. Find the non-zero eigenvalues of A^tA: $\lambda_1 > \lambda_2 > 0$.

3. Find: (i) A unit eigenvector u_1 of λ_1, a unit eigenvector u_2 for λ_2 and a unit vector u_3 such that u_1, u_2, u_3 is an orthonormal basis for $\mathbb{R}^3$.

4. Set $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, $v_1 = \frac{1}{\sigma_1}Au_1$, and $v_2 = \frac{1}{\sigma_2}Au_2$. Show that v_1, v_2 is an orthonormal basis for $\mathbb{R}^2$.

5. Let P be the orthogonal matrix whose columns are u_1, u_2, u_3, Q the orthogonal matrix whose columns are v_1, v_2, and $\sum = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix}$. Verify that $A = Q\sum P^t$.

Friday, April 12. 1. Set $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$. Find the singular value decomposition of A.

2. For $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ as in the previous problems, consider the system of equations $A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix}$. Show that this syetem of equation has no solution, and then find the best approximate solution by first calculating the pseudo-inverse A^+.

Monday, April 15. 1. For the matrix A in problem 1 from April 12, find a best approximation to the system $A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, which has no solution.

2. For the matrix A in problem 2 from April 12, calculate the SVD ins two ways, first starting with A^tA and then starting with AA^t.

Bonus Problem 5. Let C be an $s \times t$ matrix with entries in $\mathbb{R}$. Suppose $u \in \mathbb{R}^t$ is a column vector with the following property: u is in the null space of C and u^t is in the row space of C. Prove that $u = \vec{0}$. (5 points). Due at the start of class on Friday, April 19.

Wednesday, April 17. Section 5.1: 1, 3d; and Section 5.3: 1, 2, 3.

Friday, April 19. 1. Show that the matrix $A = \begin{pmatrix} 3 & 2i \\ -2i & 3 \end{pmatrix}$ is self-adjoint, and therefore normal, and then find a unitary matrix P such that P^*AP is a diagonal matrix.

2. Find the singular value decomposition for $A = \begin{pmatrix} i & 0 \\ i & i \\ 0 & i \end{pmatrix}$.

Monday, April 22. Let $A = \begin{pmatrix} 0 & 25 \\ -1 & 10 \end{pmatrix}$. Follow the steps below to arrive at the JCF for A.

(i) Find $p_A(x)$ and the eigenvalue λ of A.
(ii) Calculate E_λ.
(iii) Find a vector $v_2 \notin E_\lambda$.
(iv) Set $v_1 := (A - \lambda I)v_2$.
(v) Let P denote the 2×2 matrix with columns v_1, v_2 and find P^{-1}.
(vi) Verify that $P^{-1}AP$ is the JCF of A.

Wednesday, April 24. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}$. Follow the steps below to find the JCF of A and the change of basis matrix P.

(i) Find $p_A(x)$ and the two eigenvalues λ_1, λ_2. Arrange the eigenvalues so that λ_1 is the eigenvalue with algebraic multiplicity 2.
(ii) Calculate E_{λ_1}.
(ii) Find a vector v_2 in the null space of $(A - \lambda_1 I)^2$ that is not in E_{λ_1}.
(iv) Set $v_1 := (A - \lambda_1 I)v_2$.
(v) Take v_3 any eigenvector associated to λ_2.
(vi) Letting P be the matrix whose columns are v_1, v_2, v_3 verify that $P^{-1}AP = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$.

Friday, April 26. 1. Let $A = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 2 & -1 \\ 2 & 0 & 0 \end{pmatrix}$. Follow the steps below to find the JCF of A and the change of basis matrix P.

(i) Find $p_A(x)$ and the single eigenvalue λ.
(ii) Calculate E_λ.
(iii) Find $v_2 \notin E_\lambda$.
(iv) Set $v_1 := (A - \lambda I)v_2$. This turns out to be a vector in E_λ.
(v) Take $v_3 \in E_\lambda$ not a multiple of v_1.
(vi) Letting P be the matrix whose columns are v_1, v_2, v_3, verify that $P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Follow the steps below to find the JCF of A and the change of basis matrix P.

(i) Find $p_A(x)$ and the single eigenvalue λ.
(ii) Calculate E_λ.
(iii) Calculate $(A - \lambda I)^2$.
(iv) Find v_3 not in the null space of $(A - \lambda I)^2$.
(v) Take $v_2 := (A - \lambda I)v_3$ and $v_1 := (A - \lambda I)v_2$.
(vi) Letting P be the matrix whose v_1, v_2, v_3, verify that $P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$.

Monday, April 29. Find the JCF and change of basis matrix for each of the following matrices. Be sure to check that the change of basis product gives the correct answer. $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 6 \\ 6 & -2 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$.